Group Homomorphisms (contd.)

Zero homomorphism: The map $0 : G \to G'$ defined as $0(x) = e'$, $\forall x \in G$ is a homomorphism. It is called the zero homomorphism.

Isomorphic Groups: A group G is s.t.b. isomorphic to another G' if there is an isomorphism $f : G \longrightarrow G'$.
Notation $G \approx G'$

- Note that 'being isomorphic to' i.e. '$\approx$' is an equivalence relation on the family of groups.

- For a subgroup H of a group G, henceforth, we shall write $H \leq G$.

Theorem Let $f : G \longrightarrow G'$ be a hom. Then

i) $H \leq G \Rightarrow f(H) \leq G'$
ii) $K \leq G' \Rightarrow f^{-1}(K) \leq G$.

<u>Proof</u> Recall that $H \leq G$ iff
$x, y \in H \Rightarrow x \cdot y^{-1} \in H$.

i) Assume that $H \leq G$.
Take any $f(x), f(y) \in f(H)$, where $x, y \in H$. Since $H \leq G$, $x, y \in H \Rightarrow x \cdot y^{-1} \in H$.

$$\text{Now } f(x) \cdot (f(y))^{-1} = f(x) \cdot f(y^{-1})$$
$$= f(x \cdot y^{-1})$$
$$\in f(H) \quad (\because x \cdot y^{-1} \in H)$$

$e \in H \Rightarrow f(e) \in f(H) \quad \therefore f(H) \neq \phi$.
Thus $f(H) \leq G'$.

ii) Assume that $K \leq G'$.

~~Take any $a = f^{-1}(x), b = f^{-1}(y) \in f^{-1}(K)$,~~
~~where $x = f(a), y = f(b) \in K$ and so $x \cdot y^{-1} \in K$.~~
~~Now, $f^{-1}(x) \cdot (f^{-1}(y))^{-1}$~~
~~$= f^{-1}(f(a))$~~

Since $f(e) = e'$ and $e' \in K$,
$e \in f^{-1}(K)$. Thus, $f^{-1}(K) \neq \phi$.
Take any $x, y \in f^{-1}(K)$. Then
$f(x), f(y) \in K$. Since $K \leq G'$,
$f(x \cdot y^{-1}) = f(x) \cdot f(y^{-1}) = f(x) \cdot (f(y))^{-1} \in K$

$\therefore \; x \cdot y^{-1} \in f^{-1}(K)$

$\therefore \; f^{-1}(K) \leq G.$ $\square$

<u>Theorem</u> Let $f: G \to G'$ be a hom. Then G is abelian $\Rightarrow f(G)$ is abelian.

<u>Proof</u>. Take any $f(x), f(y) \in f(G)$, where $x, y \in G$.

Now $f(x) \cdot f(y) = f(x \cdot y)$ ($\because f$ is hom)

$= f(y \cdot x)$ ($\because G$ is abelian)

$= f(y) \cdot f(x)$ ($\because f$ is hom)

Thus $f(G)$ is abelian. $\square$

<u>Cor</u>. Let $f: G \to G'$ be a surjective hom. Then G is abelian $\Rightarrow G'$ is abelian.

<u>Proof</u>. Since f is surjective, $f(G) = G'$. $\square$