Complex Analysis (contd.)

Construction of an analytic function (Milne-Thomson's Method)

Let an analytic function $f(z) = u(x,y) + iv(x,y)$ is to be obtained as a function of z, if its real part $u(x,y)$ is given.

Here $z = x+iy$. So, $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

Hence, $f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$ —(1)

(1) may be considered as an identity in two independent variables z & $\bar{z}$.

Taking $\bar{z} = z$, we have

$f(z) = u(z,0) + iv(z,0)$

Now, $f'(z) = u_x + iv_x$

$= u_x - iu_y$, using CR equation for v_x as $u(x,y)$ is given

$= \phi_1(x,y) - i\phi_2(x,y)$, where $\phi_1(x,y) = u_x(x,y)$ and $\phi_2(x,y) = u_y(x,y)$

$= \phi_1(z,0) - i\phi_2(z,0)$

$\therefore \boxed{f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]\,dz + c}$

c being an arbitrary constt.

Similarly, if $v(x,y)$ is given, then

$$f(z) = \int [\psi_1(z,0) + i\,\psi_2(z,0)]\,dz + k$$

where $\psi_1(x,y) = v_y(x,y)$, $\psi_2(x,y) = v_x(x,y)$.

or

$$\boxed{f(z) = i\int [\eta_1(z,0) - i\eta_2(z,0)]\,dz + k}$$

where $\eta_1(x,y) = v_x(x,y)$ and $\eta_2(x,y) = v_y(x,0)$;

k being an arbitrary constant.

Ex. 1 If $u(x,y) = e^x(x\cos y - y\sin y)$, find the analytic function $f(z) = u + iv$.

Sol. ~~Sup~~ We have $u_x(x,y) = e^x(x\cos y - y\sin y) + e^x\cos y$

$$u_y(x,y) = e^x[-x\sin y - \sin y - y\cos y]$$

$$\therefore \phi_1(z,0) = u_x(z,0) = e^z(z+1)$$

$$\phi_2(z,0) = u_y(z,0) = e^z\cdot 0 = 0$$

$\therefore$ By Milne Thomson method

$$f(z) = \int [\phi_1 - i\phi_2]\,dz + c$$

$$= \int e^z(z+1)\,dz + c$$

$= e^{z}(z+1) - e^{z} + c$

$f(z) = z e^{z} + c$, c being an arbitrary constant.

□ SKS.

Ex. 2 If $v(x,y) = 3x^2y - 6xy + 3y - y^3$,

find the analytic function $f(z) = u + iv$.

Sol.

$v_x(x,y) = 6xy - 6y \Rightarrow v_x(z,0) = 0 = \eta_1(z,0)$

$v_y(x,y) = 3x^2 - 6x + 3 - 3y^2$

$\Rightarrow v_y(z,0) = 3z^2 - 6z + 3 = \eta_2(z,0)$.

By Milne-Thomson method

$$f(z) = i\int [\eta_1(z,0) - i\,\eta_2(z,0)]\,dz + k.$$

$$= i\int \{0 - i(3z^2 - 6z + 3)\}\,dz + k$$

$$f(z) = z^3 - 3z^2 + 3z + k$$ □ SKS.