Linear sum and Direct sum of two subspaces of a V.S. —

<u>Question</u>. What is the No. of elements in a V.s. over an infinite ~~field~~ field ?

<u>Ans</u> — Either zero space or ~~are~~ infinite V.S.

<u>Def^n</u> Let W_1 and W_2 be two subspaces of a V.S. $V(F)$

$$\underbrace{W_1+W_2}_{} \overset{\text{def}}{=} \{a_1+a_2 : a_1\in W_1,\ a_2\in W_2\} \subseteq V$$

↳ <u>linear sum</u> of W_1 & W_2

<u>Claim</u>:— <u>W_1+W_2 is a subspace of V ?</u>

For, Let $a_1+a_2,\ b_1+b_2 \in W_1+W_2$ & $\alpha, \beta \in F$

$\Longrightarrow a_1, b_1 \in W_1,\ a_2, b_2 \in W_2$ & $\alpha, \beta \in F$

$\Longrightarrow \alpha a_1+\beta b_1 \in W_1,\ \alpha a_2+\beta b_2 \in W_2$ ($\because$ W_1 & W_2 are subspaces)

$\Longrightarrow (\alpha a_1+\beta b_1)+(\alpha a_2+\beta b_2) \in W_1+W_2$ (from def^n of lin. sum)

$\Longrightarrow (\alpha a_1+\alpha a_2)+(\beta b_1+\beta b_2) \in W_1+W_2$

($\because$ + is comm. & Associative)

$$\Rightarrow \alpha(a_1+a_2)+\beta(b_1+b_2) \in W_1+W_2$$

$\therefore$ $\cancel{W_1 \oplus W_2}$ W_1+W_2 is subspace of V.S

Example

$V = \mathbb{R}^3(\mathbb{R})$

$\cancel{W_1 = \{(x,y,z) : x \in}$

$W_1 = \{(0,y,z) : y,z \in \mathbb{R}\}$

$W_2 = \{(x,0,z) : x,z \in \mathbb{R}\}$

$W_1 + W_2 = ?$

Let $(x,y,z) \in \mathbb{R}^3$

$$\because (x,y,z) = \underbrace{\left(0,y,\tfrac{z}{2}\right)}_{\in W_1} + \underbrace{\left(x,0,\tfrac{z}{2}\right)}_{\in W_2} \in W_1+W_2$$

$\mathbb{R}^3 = W_1 + W_2$ $\qquad$ $\mathbb{R}^3 \neq W_1 \oplus W_2$

$W_3 = \{(x,0,0) : x \in \mathbb{R}\}$

$W_1 = \{(0,y,z) : y,z \in \mathbb{R}\}$

$W_1 + W_2 \subseteq \mathbb{R}^3$

Let $(x,y,z) \in \mathbb{R}^3$

$$\because (x,y,z) = (x,0,0) + (0,y,z) \in W_3 + W_1$$

unique linear combination.

$\mathbb{R}^3 = W_1 + W_3$

$= W_1 \oplus W_2$

Defⁿ Direct sum :- A v.s V over a field F is said to be the direct sum of its two subspaces W_1 & W_2 iff each element in V can be uniquely written as one element of Ist subspace + one element of second subspace.

i.e V is direct sum of its two subspaces W_1 & W_2 iff

$x \in V$ can be uniquely written as

$x = a_1 + a_2 : a_1 \in W_1, a_2 \in W_2$

and we write

$V = W_1 \oplus W_2$