

Group Theory (contd.)

Recall that $H \trianglelefteq G$ iff $g \cdot H = H \cdot g, \forall g \in G$.

i.e. iff Left coset of H

= right coset of H

~~i.e. iff $[G:H] = [G:H]$~~

Theorem A subgroup of index 2 is normal.

or

Let G be a group and let $H \leq G$ with $[G:H] = 2$. Then $H \trianglelefteq G$.

Proof. Take any $g \in G$.

Case-I If $g \in H$, we have

$$g \cdot H = H = H \cdot g$$

$$\Rightarrow H \trianglelefteq G.$$

Case-II If $g \notin H$, $g \cdot H \neq H = e \cdot H$

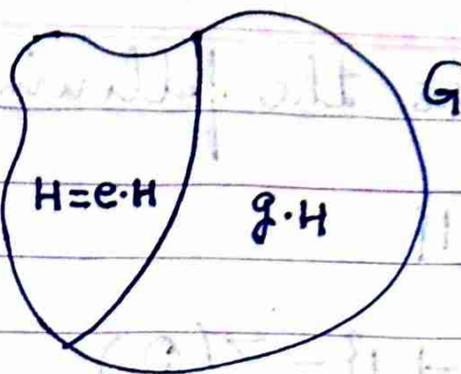
$$\Rightarrow g \cdot H \cap e \cdot H = \emptyset$$

$$\text{and } H \cdot g \neq H = H \cdot e \Rightarrow H \cdot g \cap H \cdot e = \emptyset$$

Since $[G:H] = 2$,

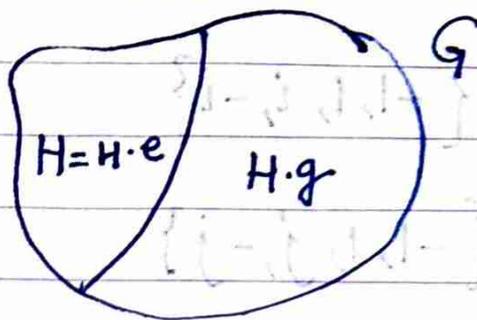
$$G = g \cdot H \cup e \cdot H$$

$$\begin{aligned} \Rightarrow g \cdot H &= G \setminus e \cdot H \\ &= G \setminus H \quad \text{--- (1)} \\ (\because g \cdot H \cap e \cdot H &= \emptyset) \end{aligned}$$



Also,

$$G = H \cdot g \cup H \cdot e$$



$$\begin{aligned} \Rightarrow H \cdot g &= G \setminus H \cdot e \\ &= G \setminus H \quad \text{--- (2)} \\ (\because H \cdot g \cap H \cdot e &= \emptyset) \end{aligned}$$

By (1) & (2), we have $g \cdot H = H \cdot g, \forall g \in G$.
 $\therefore H \trianglelefteq G$.

□ SKS

Note that for a finite group G ,
 $[G:H] = \frac{o(G)}{o(H)}$.

Also, note that a subgroup of an abelian group is normal, i.e.

If G is abelian, then all subgroups of G are normal.

Converse of this result is however not true, e.g. consider the Quaternion group
 $\mathbb{Q}_8 = \{1, i, j, k, -i, -j, -k \mid i \cdot j = -j \cdot i = k, j \cdot k = -k \cdot j = i, k \cdot i = -i \cdot k = j\}$.

\mathbb{Q}_8 has the following subgroups:

$$H_1 = \{1\}$$

$$H_2 = \{-1, 1\} = Z(\mathbb{Q}_8)$$

$$H_3 = \{-1, 1, i, -i\}$$

$$H_4 = \{-1, 1, j, -j\}$$

$$H_5 = \{-1, 1, k, -k\}$$

$$H_6 = \mathbb{Q}_8$$

H_1 and H_6 are trivially normal (improper subgroups).

$$H_2 = Z(\mathbb{Q}_8) \trianglelefteq \mathbb{Q}_8$$

$$[\mathbb{Q}_8 : H_3] = \frac{o(\mathbb{Q}_8)}{o(H_3)} = \frac{8}{4} = 2 \quad \text{in } H_3 \trianglelefteq \mathbb{Q}_8$$

$$\text{Similarly, } [\mathbb{Q}_8 : H_4] = [\mathbb{Q}_8 : H_5] = 2.$$

$$\text{So, } H_4 \trianglelefteq \mathbb{Q}_8, H_5 \trianglelefteq \mathbb{Q}_8.$$

Thus, all subgroups of \mathbb{Q}_8 are normal in it. But, \mathbb{Q}_8 is a non-abelian group.

□ SKS.

Quotient group or Factor group

Let G be a group and let $H \trianglelefteq G$.

Define $G/H = \{ \text{set of all left cosets of } H \}$

$$= \{ g \cdot H \mid g \in G \}$$

N.B. G/H may also be defined as the set of all right cosets.

Since $H \trianglelefteq G$, $g \cdot H = H \cdot g$,
 $\forall g \in G$

This set forms a group w.r.t. the binary operation '*' defined as

$$(r \cdot H) * (s \cdot H) = (r \cdot s) \cdot H$$

We check all group axioms as under:

i) Associativity

$$\left((r \cdot H) * (s \cdot H) \right) * (t \cdot H) = \left((r \cdot s) \cdot H \right) * (t \cdot H)$$

$$= \left((r \cdot s) \cdot t \right) \cdot H$$

$$= (r \cdot (s \cdot t)) \cdot H \quad (\because r, s, t \in G)$$

$$= r \cdot H * (s \cdot t) \cdot H$$

$$= r \cdot H * \left((s \cdot H) * (t \cdot H) \right)$$

ii) Existence of identity

For all $g \cdot H \in G/H$, $\exists e \cdot H = H \in G/H$

$$\text{s.t. } (g \cdot H) * (e \cdot H) = (g \cdot e) \cdot H = g \cdot H$$

$$\text{and } (e \cdot H) * (g \cdot H) = (e \cdot g) \cdot H = g \cdot H$$

So, H is the identity element of G/H .

iii) Existence of inverse

For each $g \cdot H \in G/H$, $\exists g^{-1} \cdot H \in G/H$

($\because g \in G \Rightarrow g^{-1} \in G$ exists) s.t.

$$(g \cdot H) * (g^{-1} \cdot H) = (g \cdot g^{-1}) \cdot H = e \cdot H = H$$

$$\text{and } (g^{-1} \cdot H) * (g \cdot H) = (g^{-1} \cdot g) \cdot H = e \cdot H = H$$

So, $g^{-1} \cdot H$ is the inverse of $g \cdot H$ in G/H .

Thus, $(G/H, *)$ is a group.

This group is called the Quotient group G by H or the factor group

of G by H . \square s.k.s. $\&$

N.B. If G is abelian, G/H is also abelian.