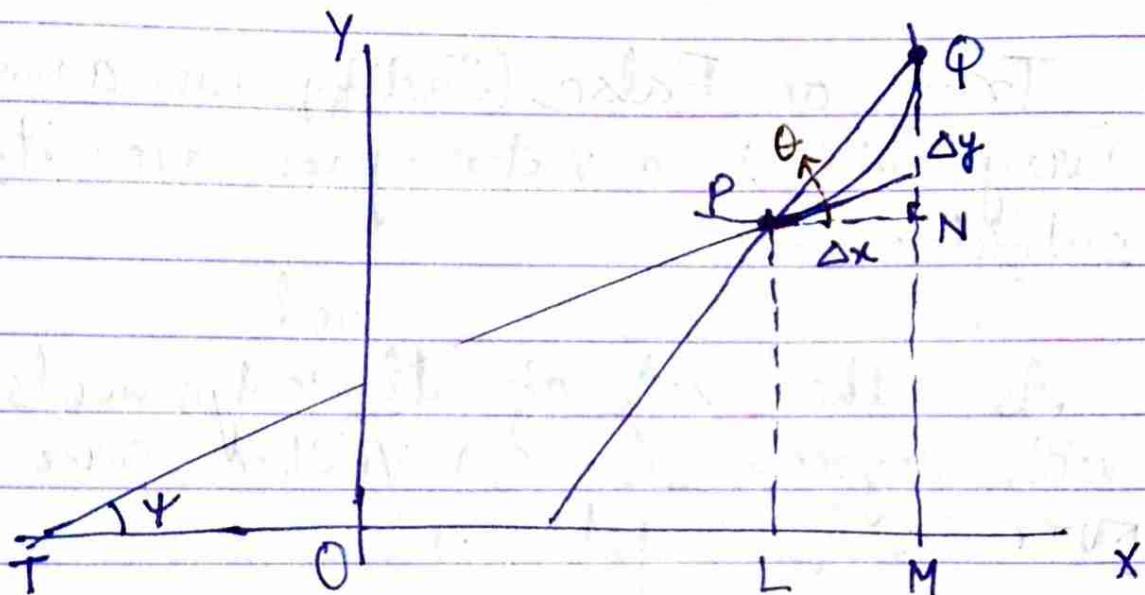


# Differential Calculus (Contd.)

## Tangents and Normal

- Geometrical interpretation of derivative at a point  $P(x, y)$  of the curve  $y = f(x)$ :



Let  $Q(x + \Delta x, y + \Delta y)$  be a neighbouring point. Please see the above figure.

$$\tan \theta = \frac{\Delta y}{\Delta x}$$

Observe that as  $Q \rightarrow P$ ,  $\Delta x \rightarrow 0$ .

$$\text{So, } \lim_{Q \rightarrow P} (\tan \theta) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \quad (1)$$

Also, as  $Q \rightarrow P$ , chord PQ becomes the tangent TP at P and so  $\theta \rightarrow \psi$ . Thus

$$\lim_{Q \rightarrow P} (\tan \theta) = \boxed{\tan \psi = \frac{dy}{dx}}, \text{ from (1).}$$

But  $\tan \psi$  is the slope of tangent TP.

Thus,  $\frac{dy}{dx}$  at  $P(x, y)$  is the slope of

the tangent TP to the curve  $y=f(x)$  at  $P(x, y)$ .

- Equation of tangent and normal at a point  $P(x, y)$  to the curve  $y=f(x)$ .

Equation of the tangent TP at  $P(x, y)$

$$Y-y = m(x-x), \text{ where } m = \tan \psi.$$

$$\text{But, } m = \tan \psi = \left. \frac{dy}{dx} \right|_{\text{at } P}.$$

So, the equation of TP is

$$Y-y = \left. \frac{dy}{dx} \right|_{\text{at } P} (x-x)$$

Suppose the equation of the normal at P is

$$Y-y = m_1(x-x), \text{ where}$$

$m_1$  is the slope of the normal at P.

$$\text{So, } m \cdot m_1 = -1 \text{ or } m_1 = -\frac{1}{m} = -\frac{1}{\frac{dy}{dx}}$$

$\therefore$  Equation of the normal at P becomes

$$Y-y = -\frac{1}{(dy/dx)}_{P_0} (X-x)$$

□ SKS.

Ex. Find the equations of the tangent and normal to the curve

$y(x-2)(x-3) - x + 7 = 0$  at the point where it meets the axis of  $x$ .

Sol. The curve  $y(x-2)(x-3) - x + 7 = 0$  — ①  
meets  $x$ -axis at  $P(7, 0)$ .

Differentiating (1) wrt 'x', we get

$$\frac{dy}{dx} (x-2)(x-3) + y[1](x-3) + (x-2)\cdot 1 - 1 = 0$$

At  $P(7, 0)$ , we have

$$\left. \frac{dy}{dx} \right|_{P(7,0)} \cdot (7-2)(7-3) - 1 = 0$$

$$\Rightarrow \left. \frac{dy}{dx} \right|_{P(7,0)} = \frac{1}{20}$$

∴ Tangent at  $P(7, 0)$ :

$$y-0 = \left. \frac{dy}{dx} \right|_{P_0} (x-7)$$

$$y = \frac{1}{20} (x-7) \Rightarrow x - 20y = 7$$

Normal at  $P(7, 0)$  is

$$y - 0 = - \frac{1}{(dy/dx)_P} (x - 7)$$

$$\Rightarrow y = -20(x - 7)$$

$$\Rightarrow 20x + y = 140. \quad \square \text{ SKS}$$

Exercise 1. Show that the tangent and the normal at any point of the curve

$$x = a e^\theta (\sin \theta - \cos \theta)$$
$$y = a e^\theta (\sin \theta + \cos \theta)$$

are equidistant from the origin.

(Hint:  $\left. \frac{dy}{dx} \right|_{\text{at } \theta} = - \frac{dy/d\theta}{dx/d\theta}$ )

2. Find the condition for the curves  $ax^2 + by^2 = 1$ ,  $a'x^2 + b'y^2 = 1$  to intersect orthogonally.