Qu. Discuss Carnot's reversible heat engine. Show how the work done in each operation is represented on a pressure volume diagram. Deduce an expression for the efficiency of Carnot's reversible heat engine.

Ans. A heat engine is a practical arrangement of converting heat energy into mechanical work. In practice, the efficiency of a heat engine is very small and ranges between $5 \rightarrow 25\,\%$. This shows that maximum of 25% heat energy supplied to machine is converted into mechanical work.

The function of the steam engine is to convert the chemical energy stored in coal to energy of motion by utilizing principles involving in the running of a heat engine based on three factors.

(i) Source of heat
(ii) On working substance and
(iii) Suitable machinery.

In addition of these three the fourth is temperature difference Carnot designed an ideal heat engine which, of course, cannot be realized in actual practice but has a maximum efficiency. It consists of

(a) A cylinder having perfectly non-conducting walls and a perfectly conducting bottom.
(b) A hot body to serve as a source of heat at a constant temperature T_1 Kelvin.
(c) A cold body to serve as a sink at a constant lower temperature T_2 Kelvin.

The source and the sink are large reservoirs of heat so that their temperature remain practically unchanged during any transfer of heat to or from the cylinder. The process occurring in the engine are reversible. The behavior of the working gas is shown by the indicator diagram showing the pressure and volume of the gas at any instant. The reversible cyclic process consists of a sequence of isothermal and adiabatic curves on a P-V diagram and is known as Carnot cycle.

In a Carnot cycle the working substance is supposed to undergo the following four quasi static operations.

Expansion

(i) Isothermal expansion
(ii) Adiabatic expansion

Compression

(i) Isothermal compression
(ii) Adiabatic compression.

P-V diagram

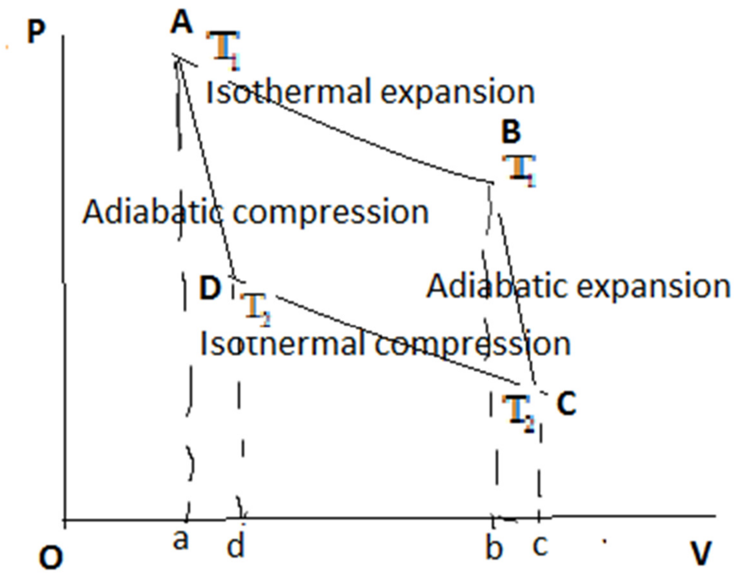

Fig. 1 Figure shows the Carnot cycle at P- V diagram.

(i) Adiabatic expansion:

Let us consider a system working with suitable machinery at reservoir temperature T_1. The piston of the system moves outward very slowly so that the gas expand quasi-statically. As the gas expand the temperature tends to fall. Heat is, therefore, absorbed from the source at a constant temperature T_1. The change is isothermal and represented by the line AB at the P-V diagram. The change is isothermal, there is no change in the internal energy of the gas. According to 1st law of thermodynamics

$$dQ = dU + PdV \qquad (1)$$

For isothermal changes $dU = 0$, hence Eq. (1) reduces to

$dQ = PdV$, and for path A $\rightarrow$ B

$\int_A^B dQ = \int_A^B PdV$, and

$$W_1 = \int_A^B PdV$$

$$= RT_1 \int_a^b \frac{dV}{V}$$

$$W_1 = R\, T_1 \ln \frac{V_b}{V_a} \qquad (2)$$

(ii) Adiabatic expansion

Now remove the cylinder from source and place it on the perfectly insulated stand and allow the gas in the system to expand adiabatically till the temperature falls to T_2 The change is represented by the adiabatic path BC. According to 1st law of thermodynamics

$$dQ = dU + PdV \quad .$$

For adiabatic changes, $dQ = 0$, then

$$dU = - PdV \, ,$$

and for path B $\rightarrow$ C

$$W_2 = \int_B^C dU = - \int_B^C PdV$$

$$W_2 = - \int_B^C PdV$$

$$W_2 = - \int_b^c V^{-\gamma} \, dV$$

$$= - \frac{1}{1-\gamma} [P_C V_C - P_b V_b]$$

$$W_2 = - \frac{R(T_2 - T_1)}{1-\gamma} \qquad (3)$$

(iii) Isothermal compression

In isothermal compression the cylinder is placed in contact with the sink at temperature T_2.
This is equal to work done on the gas. According to 1st law of the thermodynamics

$$dQ = dU + pdV \, .$$

For isothermal compression, dU = 0, then

$dQ = PdV$ and for complete path C $\rightarrow$ D

$$W_3 = \int_C^D dQ = \int_C^D PdV = -\int_d^c PdV = -RT_2 \ln\frac{V_c}{Vd} \text{ i.e}$$

$$W_3 = -RT_2 \ln\frac{V_c}{V_d} \tag{4}$$

(iv) Adiabatic compression

In adiabatic compression the cylinder is now placed in contact with heat reservoir and gas is compressed adiabatically. The work done on the gas by adiabatic compression is calculated from 1st law of thermodynamics

$$dQ = dU + pdV$$

For adiabatic dQ =0, hence

$$W_4 = \int_d^a dU = -\int_d^a PdV = -\frac{1}{1-\gamma}[P_a V_a - P_d V_d] = +\frac{R}{1-\gamma}[T_2 - T_1]$$

$$W_4 = \frac{R}{1-\gamma}[T_2 - T_1]$$

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The net work done

The net work done by the engine in per cycle is W, then

$$W = W_1 + W_2 + W_3 + W_4$$

Putting the value from Eq. 2,Eq.3, Eq.4, Eq.5, we get

$$W = RT_1 \ln\frac{V_b}{V_a} - R\frac{(T_2 - T_1)}{1-\gamma} - RT_2 \ln\frac{V_c}{V_d} + R\frac{(T_2 - T_1)}{1-\gamma} \quad .$$

The W_2 and W_4 opposite in sign hence cancelled, the net work

$$W = RT_1 \ln\frac{V_b}{V_a} - RT_2 \ln\frac{V_c}{V_d} \tag{6}$$

In fig. ABCD is the closed cycle in which point b and c at the same adiabatic path. Therefore

$$T_1 V_b^{\gamma-1} = T_2 V_c^{\gamma-1}$$

or

$$\frac{T_1}{T_2} = \frac{V_c^{\gamma-1}}{V_b^{\gamma-1}}$$

or

$$\left(\frac{T_1}{T_2}\right)^{\frac{1}{1-\gamma}} = \frac{V_c}{V_b}$$

Similarly point A and D lie at same adiabatic path, so that

$$\left(\frac{T_1}{T_2}\right)^{\frac{1}{1-\gamma}} = \frac{V_d}{V_a} = \rho = \text{adiabatic expansion ratio}$$

Therefore

$$\frac{V_c}{V_b} = \frac{V_d}{V_a}$$

or

$$\frac{V_c}{V_d} = \frac{V_b}{V_a} = r = \text{Isothermal ratio}$$

$$Q_1 = RT_1 \ln r$$

$$\frac{Q_1}{T_1} = R \ln r$$

and

$$Q_2 = RT_2 \ln r$$

$$\frac{Q_2}{T_2} = R \ln r$$

Now,

$$W = Q_1 - Q_2$$

$$W = (T_1 - T_2) R \ln r$$

or

$$\frac{W}{(T_1 - T_2)} = R \ln r = \frac{Q_1}{T_1} = \frac{Q_2}{T_2} \qquad (7)$$

From Eq. (7) we find out the efficiency of the reversible engine

$$\eta = \frac{W}{Q_1} = \frac{T_1 - T_2}{T_1}$$

$$\eta = 1 - \frac{T_2}{T_1} \qquad (8)$$

Having analyzed the mode of operation of the Carnot cycle, the following points are considerable.

(i) It is reversible at each stage.

(ii) No engine can be more efficient than the Carnot engine.

(iii) The nature of the working substance is immaterial

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